Some Analyses of Wave Shapes Used in Harmonic Producers

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Analyses by Fourier's Series have been made of waves consisting of sinusoidal, rectangular and trapezoidal pulses and also waves of the type found in multivibrator circuits. The method of increasing harmonic content by modulating a wave with a submultiple is treated mathematically.

THE heterodyne method of frequency comparison requires, except in the case of the comparison of nearly identical frequencies, the generation of harmonics of either the unknown, or of the standard frequency or of botb. These barmonics may be generated directly in tbe modulator which produces tbe difference frequency, or "beat note", or may be generated in an entirely separate circuit before the frequency is applied to the modulator. An example of the latter is the multivibrator circuit often used in connection witb a frequency standard to produce a series of harmonics of this standard frequency.

The design of harmonic generators for frequency measuring equipment presents a different problem from the design of equipment for producing a single harmonic such as doubler or tripler stage in a radio transmitter. In the latter case the amplitude of the one harmonic and the efficiency are of primary importance. In frequency measuring equipment, although a large amplitude of each harmonic is desirable, it is of greater importance that each harmonic within the range to be used, which may be up to the 100th or 150tb harmonic or even higher, be present and that the amplitude of nearby harmonics be of tbe same order of magnitude. Unless the latter conditions are met, there is a danger that the beats obtained with a weak harmonic will either be entirely overlooked or mistaken for a higber order modulation product.

The generation of harmonics is usually accomplished by the distortion of tbe wave shape in some nonlinear circuit element such as a vacuum tube. One such harmonic generator consists of a vacuum tube biased so that there is no output for a portion of the cycle. The plate current of such a tube may be approximated by a sine wave shaped pulse such as shown in Fig. 1. Any such periodic wave can be resolved into its harmonic components[1] and in the case of this wave the amplitude of the nth harmonic is found to be

[1] This and the subsequent analyses were made by application of Fourier's Series. See I. S. Sokolnikoff and E. S. Sokolnikoff, "Higher Mathematics for Engineers and Physicists," Chapter VI.

$$h_n = \frac{A}{n\pi(1 - \cos b/2)} \left[\frac{\sin (n - 1)b/2}{n - 1} - \frac{\sin (n + 1)b/2}{n + 1} \right] \qquad (1)$$

in which A is the amplitude of the pulse and b the pulse width as shown in Fig. 1.

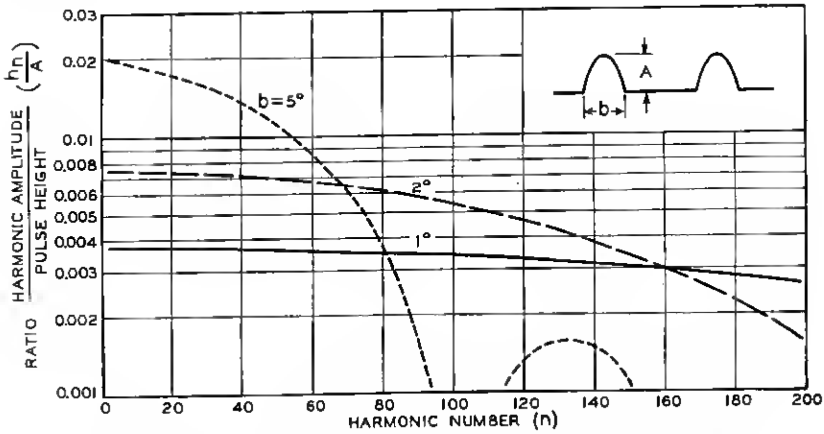

Fig. 1—Harmonic content of a wave consisting of sinusoidal pulses

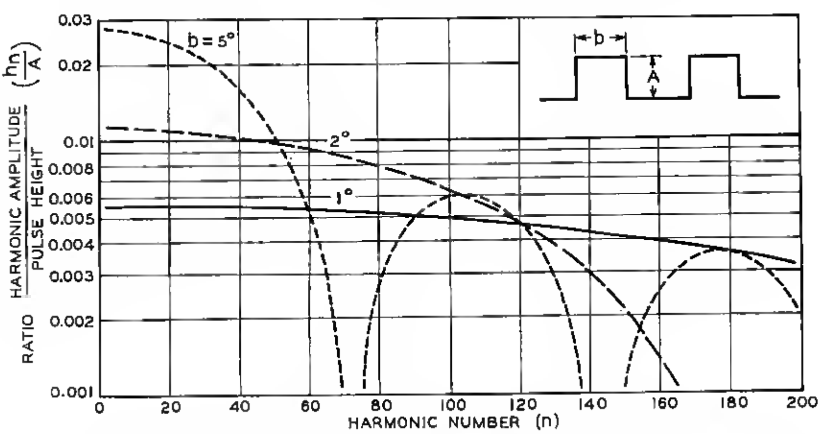

Fig. 2—Harmonic content of a wave consisting of rectangular pulses

The form of this expression immediately suggests that for some harmonics the terms

$$\frac{\sin (n - 1)b/2}{n - 1} - \frac{\sin (n + 1)b/2}{n + 1}$$

may become equal to zero causing these harmonics to vanish. That this

is the case is shown in the curves of Fig. 1 in which the harmonic amplitudes are plotted against n for pulse widths of 5°, 2° and 1°. With a 5° pulse harmonics in the vicinity of the 105th and again the 150th hecome negligihly small. For a shorter pulse width the amplitude of the lower harmonics decreases hut all harmonics up to heyond the 200th are present.

The wave shown in Fig. 1 can only be considered as a first approximation of the plate current in such a harmonic generator as it implicitly assumes that the tube is linear to cut-off. More frequently sufficient excitation is placed on the grid of the tuhe to saturate it and the resulting current wave may better be represented by a series of rectangular pulses such as shown

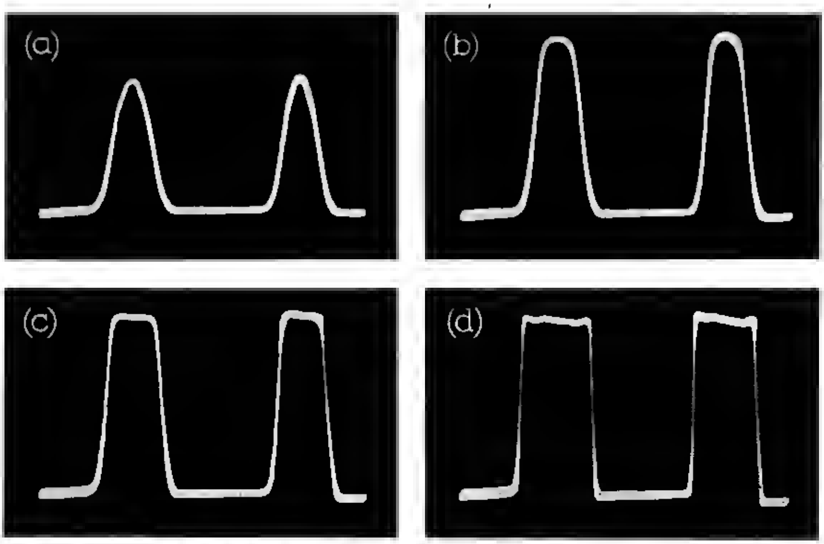

Fig. 3—Oscillograms of the plate current of a vacuum tuhe showing the transition from sinusoidal to rectangular pulses as excitation is increased

(a) Excitation 6 volts
(h) Excitation 8 volts
(c) Excitation 10 volts
(d) Excitation 20 volts

in Fig. 2. This transition from sine wave pulses to rectangular pulses as the grid excitation is increased is shown in the series of oscillographs in Fig. 3.

The analysis of a wave consisting of rectangular pulses such as the one in Fig. 2 shows the amplitude of the nth harmonic to be

$$h_n = \frac{2A}{n\pi} \sin \frac{nb}{2} \qquad (2)$$

From this equation it is seen that certain of the harmonics are not present as the expression (2) hecomes equal to zero whenever

$$n = \frac{2\pi}{b} m \tag{3}$$

$$m = 1, 2, 3, 4, \cdots$$

Thus for a rectangular pulse of 5° ($\pi/36$ radians) pulse width the 72nd, 144th, 216th, etc. harmonics vanish, and harmonics in the vicinity of these missing harmonics have lower amplitudes as can be seen from the curves of Fig. 2.

As the pulse width of a rectangular wave increases, the number of harmonics which vanish increases. For a pulse width of 90° every fourth harmonic is missing. For a pulse width of 180°, the familiar square wave,

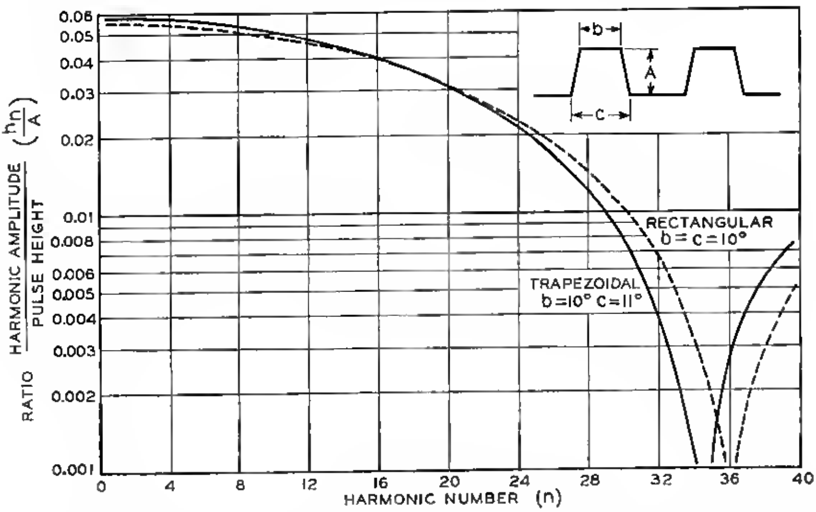

Fig. 4—Comparison of the harmonic content of waves consisting of rectangular and o trapezoidal pulses

every even harmonic vanishes and the wave contains only odd harmonics. As the pulse width is increased beyond 180° the number of harmonics increases and it can be shown that a wave having a pulse width greater than 180° will have the same harmonic content[2] as a wave of pulse width (360°—b). Thus for a large harmonic content it is desirable to have a wave having either extremely narrow pulses or pulses lasting nearly 360°.

True rectangular pulses are never obtained in practice. One common type of distortion in such pulses when obtained by the "limiter" action of a vacuum tube consists in the pulses having sloping rather than vertical sides. The sloping sides arise from the fact that the pulses are essentially sine waves

[2] This statement is correct for absolute magnitude of the harmonics only. Certain of the harmonics in the two waves will be 180° out of phase.

with their tops chopped off. The analysis of a pulse of the dimensions shown in Fig. 4 shows that the amplitude of the nth harmonics is given hy the expression

$$h_n = \frac{4A}{n^2\pi(c - b)}\left[\cos\frac{nb}{2} - \cos\frac{nc}{2}\right] \tag{4}$$

In order to show hetter the relationship hetween a wave of rectangular pulses and one of trapezoidal pulses, consider the ratio of the nth harmonic for these two waves. From (2) and (4)

$$\frac{h_n \text{ for trap. pulse}}{h_n \text{ for rect. pulse}} = \frac{2(\cos nb/2 - \cos nc/2)}{n(c - b)\sin nb/2} \tag{5}$$

Suhstituting $c - b = \delta$ and expanding $\cos nc/2 = \cos\ (nb/2 + n\delta/2)$, the right hand side of (5) becomes

$$\frac{2}{n\delta}\left[\frac{\cos nb/2}{\sin nb/2} - \frac{\cos nb/2 \cos n\delta/2}{\sin nb/2} + \sin n\delta/2\right] \tag{6}$$

For small values of $n\delta/2$, that is for trapezoidal waves whose hase is only slightly wider than the top, $\cos n\delta/2$ may he replaced by unity and $\sin n\delta/2$ hy $n\delta/2$. The first two terms then cancel and the approximation

$$\frac{h_n \text{ for trap. pulse}}{h_n \text{ for rect. pulse}} \cong 1 \tag{7}$$

is ohtained showing that a slight slope in the sides of the pulse has only a second order effect on the harmonic content of the wave.

The curve in Fig. 4 shows the harmonic content of a rectangular wave having a pulse width of 10° compared with that of a trapezoidal wave having a pulse width of 10° at the top and 11° at the bottom. For lower harmonics the amplitudes are nearly the same, hut in the vicinity of the 36th harmonic there is an essential difference. For the rectangular pulse, the 36th harmonic vanishes, while the trapezoidal pulse has a minimum at a somewhat lower value of n and all harmonics have finite values.[3] This is shown in Tahle 1 which tahulates the amplitude of the harmonics in this case.

A second form of distortion in rectangular pulses is the rounding of the corners at hoth the top and the bottoms of the pulse. This type of distortion is more difficult to analyze and while no complete analysis has heen made the effect of such distortion is known to he, in general, to reduce the amplitude of the higher harmonics.

[3] In discussing the curves in Fig. 1 thru 5 it must be remembered that while these are drawn as solid lines, the lines have a meaning only for integral values of n. Fractional values of n are meaningless.

From the examination of these cases it is evident that in the design of a harmonic generator of the type here considered the decision as to the pulse

TABLE 1
HARMONIC CONTENT OF RECTANGULAR AND TRAPEZOIDAL PULSES SHOWN IN FIGURE 4

Harmonic	$\frac{\text{Harmonic amplitude}}{\text{Pulse height}} = \frac{h_n}{A}$	
	Rectangular	Trapezoidal
Fundamental	.0555	.0581
2	.0554	.0580
3	.0550	.0576
4	.0545	.0570
5	.0540	.0563
10	.0488	.0505
15	.0411	.0416
20	.0314	.0307
25	.0209	.0191
30	.01061	.00811
32	.00681	.00411
33	.00501	.00226
34	.00325	.000489
35	.001901	−.001085
36	0	−.00276
37	−.00180	−.00412
38	−.00291	−.00557
40	−.00545	−.00791

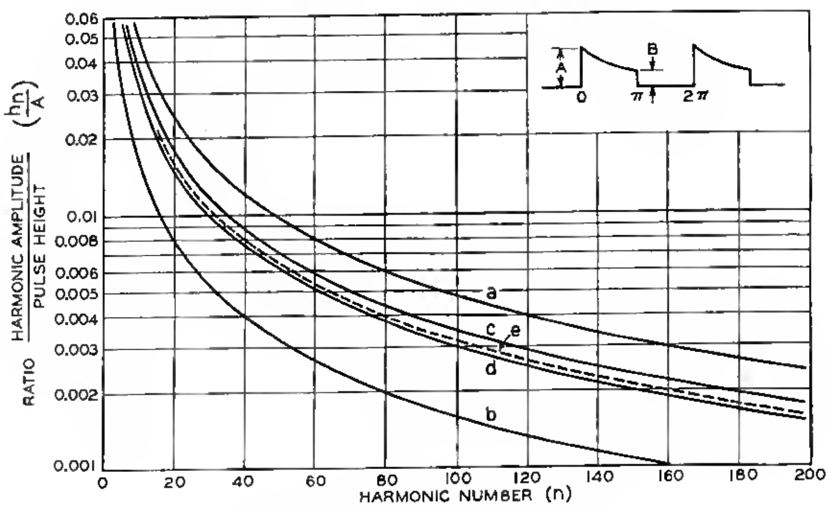

Fig. 5—Harmonic content of multivibrator wave
(a) Odd harmonics $\tau = 1/2$
(b) Even harmonics $\tau = 1/2$
(c) Odd harmonics $\tau = 1/10$
(d) Even harmonics $\tau = 1/10$
(e) All harmonics $\tau = 0$

widtb must be based on the type of service to wbicb it is to be put. If only a few harmonics are required, a considerable gain in the amplitudes of tbe harmonics can be obtained by using a wider pulse width. When a wide range of harmonics is required, the band width must be greatly reduced to avoid blank intervals in the frequency spectrum.

A second type of harmonic generator is the multivibrator. The output wave of such a harmonic generator has a shape similar to that shown in Fig. 5. The current pulse lasts for a complete 180° rising abruptly to the peak value, then falling more or less exponentially to a lower value and finally breaking abruptly to zero. Assuming an exponential decay this wave will be found to contain the following harmonics

$$h_n = \frac{A(1-\tau)}{\sqrt{n^2\pi^2 + (\ln \tau)^2}} \text{ for even harmonics} \tag{8}$$

$$h_n = \frac{A(1+\tau)}{\sqrt{n^2\pi^2 + (\ln \tau)^2}} \text{ for odd harmonics} \tag{9}$$

Except for small values of n, the $(\ln \tau)^2$ term is negligible and these equations can be written

$$h_n = \frac{A(1-\tau)}{n\pi} \text{ for even harmonics} \tag{10}$$

$$h_n = \frac{A(1+\tau)}{n\pi} \text{ for odd harmonics} \tag{11}$$

In all of the above equations $\tau = B/A$, the ratio of the amplitude at the end to the amplitude at the beginning of the pulse.

The curves in Fig. 5 show the harmonic content of such a wave for $\tau = \frac{1}{2}$ and $\tau = \frac{1}{10}$. In the first case the amplitudes of the odd and even harmonics differ by approximately 9.5 db while in the second case the amplitudes are not greatly different. The dotted curve shows the limiting condition which all harmonics approach as τ approaches zero, that is as the current at the end of the pulse approaches zero.

The analysis of such a pulse except assuming a linear rather than exponential decay yields the following equations

$$h_n = \frac{A\ (1-\tau)}{n\pi} \text{ for even harmonics} \tag{12}$$

$$h_n = \frac{A}{n\pi}\sqrt{(1+\tau)^2 + \frac{4(1-\tau)^2}{n^2\pi^2}} \text{ for odd harmonics} \tag{13}$$

As n becomes large the second term under the radical becomes small and (13) becomes

$$h_n = \frac{A\,(1 + \tau)}{n\pi} \text{ for odd harmonics} \tag{14}$$

Equations (12) and (14) are identical with (10) and (11) showing that in harmonic generators of this type the harmonic content of the output wave is primarily a function of the initial and final values of the current rather than of the shape of the decay curve.

All of the foregoing curves show that the amplitudes of the higher harmonics are quite small so that in many applications some method of increasing their amplitudes may be required. This can be accomplished by the use of tuned amplifiers. An alternative method is to modulate a standard frequency wave with a lower derived frequency.

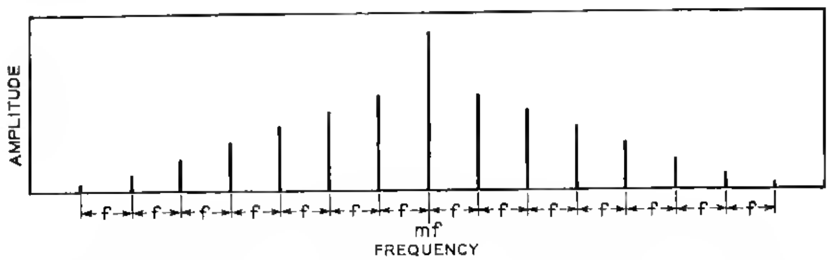

Fig. 6—Frequency spectrum of wave of frequency mf modulated by a series of pulses of frequency f

Assume a standard frequency of the form

$$A \cos m\omega t$$

This wave is completely modulated by a rectangular wave of frequency $\omega/2\pi$ and pulse width b. The modulated wave will then be of the form

$$I = A[1 + Kf(t)] \cos m\omega t \tag{15}$$

As shown previously the modulating wave is of the form

$$f(t) = \frac{b}{2\pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{nb}{2} \cos n\omega t \tag{16}$$

For 100 per cent modulation $K = 1$. Since

$$\cos (m\omega t) + \cos (n\omega t) = \tfrac{1}{2} \cos (m + n)\, \omega t + \tfrac{1}{2} \cos (m - n)\, \omega t \tag{17}$$

the modulated wave is

$$i_p = \frac{Ab}{2\pi} \cos m\omega t + \sum_{n=1}^{\infty} \frac{A}{n\pi} \sin \frac{nb}{2} \cos (m + n)\, \omega t + \sum_{n=1}^{\infty} \frac{A}{n\pi} \sin \frac{nb}{2} \cos (m - n)\, \omega t \tag{18}$$

The frequency spectrum of this wave is shown in Fig. 6. The original standard frequency $m\omega/2\pi$ is present and on either side above and below $\omega/2\pi$ cycles apart are additional components. The rate at which the amplitude of these frequencies dies out depends on the modulating pulse width and is equal to half the amplitude of the corresponding harmonic in Fig. 2.

If the standard frequency is not a pure wave but contains harmonics each of these harmonics will be modulated by the rectangular pulses, that is the function (16). The result will be a series of frequency spectra similar to the one in Fig. 6, each centered at one of the harmonics of the standard frequency. By proper choice of the frequency of the modulating wave these spectra may be made to overlap giving a continuous series of harmonic of the modulating frequency with much larger amplitudes than can be obtained from a straightforward harmonic generator. As an example, a one-megacycle wave heavily modulated with 100 kc was found to give strong 100 kc harmonics up to well over 35 mc.

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